

# ADDING A LOT OF COHEN REALS BY ADDING A FEW II

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**ABSTRACT.** We study pairs  $(V, V_1)$ ,  $V \subseteq V_1$ , of models of  $ZFC$  such that adding  $\kappa$ -many Cohen reals over  $V_1$  adds  $\lambda$ -many Cohen reals over  $V$  for some  $\lambda > \kappa$ .

## 1. INTRODUCTION

We continue our study from [3]. We study pairs  $(V, V_1)$ ,  $V \subseteq V_1$ , of models of  $ZFC$  with the same ordinals, such that adding  $\kappa$ -many Cohen reals over  $V_1$  adds  $\lambda$ -many Cohen reals over  $V$  for some  $\lambda > \kappa$ <sup>1</sup>. We are mainly interested when  $V$  and  $V_1$  have the same cardinals and reals. We prove that for such models, adding  $\kappa$ -many Cohen reals over  $V_1$  cannot produce more Cohen reals over  $V$  for  $\kappa$  below the first fixed point of the  $\aleph$ -function, but the situation at the first fixed point of the  $\aleph$ -function is different. We also reduce the large cardinal assumptions from [1, 3] to the optimal ones.

## 2. ADDING MANY COHEN REALS BY ADDING A FEW: A GENERAL RESULT

In this section we prove the following general result.

**Theorem 2.1.** *Suppose  $\kappa < \lambda$  are infinite (regular or singular) cardinals, and let  $V_1$  be an extension of  $V$ . Suppose that in  $V_1$ :*

- (a)  $\kappa < \lambda$  are still infinite cardinals<sup>2</sup>,
- (b) there exists an increasing sequence  $\langle \kappa_n : n < \omega \rangle$  of regular cardinals, cofinal in  $\kappa$ . In particular  $\text{cf}(\kappa) = \omega$ ,

- (c) there is an increasing (mod finite) sequence  $\langle f_\alpha : \alpha < \lambda \rangle$  of functions in the product

$$\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n),$$

- (d) there is a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\lambda$  into sets of size  $\lambda$  such that for every countable

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<sup>1</sup>By “ $\lambda$ -many Cohen reals” we mean a generic object  $\langle s_\alpha : \alpha < \lambda \rangle$  for the poset  $\mathbb{C}(\lambda)$  of finite partial functions from  $\lambda \times \omega$  to  $2$ ”.

<sup>2</sup> $\lambda$  can be a regular or a singular cardinal, but by (b),  $\kappa$  is necessarily a singular cardinal of cofinality  $\omega$ .

set  $I \in V$  and every  $\sigma < \kappa$  we have  $|I \cap S_\sigma| < \aleph_0$ .

Then adding  $\kappa$ -many Cohen reals over  $V_1$  produces  $\lambda$ -many Cohen reals over  $V$ .

**Remark 2.2.** Condition (c) holds automatically for  $\lambda = \kappa^+$ ; given any collection  $\mathcal{F}$  of  $\kappa$ -many elements of  $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$ , there exists  $f$  such that for each  $g \in \mathcal{F}$ ,  $f(n) > g(n)$  for all large  $n$ <sup>3</sup>. Thus we can define by induction on  $\alpha < \kappa^+$ , an increasing (mod finite) sequence  $\langle f_\alpha : \alpha < \kappa^+ \rangle$  in  $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$ <sup>4</sup>.

*Proof.* Force to add  $\kappa$ -many Cohen reals over  $V_1$ . Split them into  $\langle r_{i,\sigma} : i, \sigma < \kappa \rangle$  and  $\langle r'_\sigma : \sigma < \kappa \rangle$ . Also in  $V$ , split  $\kappa$  into  $\kappa$ -blocks  $B_\sigma, \sigma < \kappa$ , each of size  $\kappa$ , and let  $\langle f_\alpha : \alpha < \lambda \rangle \in V_1$  be an increasing (mod finite) sequence in  $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$ . Let  $\alpha < \lambda$ . We define a real  $s_\alpha$  as follows. Pick  $\sigma < \kappa$  such that  $\alpha \in S_\sigma$ . Let  $k_\alpha = \min\{k < \omega : r'_\sigma(k)\} = 1$  and set

$$\forall n < \omega, s_\alpha(n) = r_{f_\alpha(n+k_\alpha), \sigma}(0).$$

The following lemma completes the proof.

**Lemma 2.3.**  $\langle s_\alpha : \alpha < \lambda \rangle$  is a sequence of  $\lambda$ -many Cohen reals over  $V$ .

**Notation 2.4.** (a) For a forcing notion  $\mathbb{P}$  and  $p, q \in \mathbb{P}$ , we let  $p \leq q$  mean  $p$  is stronger than  $q$ .

(b) For each set  $I$ , let  $\mathbb{C}(I)$  be the Cohen forcing notion for adding  $I$ -many Cohen reals. Thus  $\mathbb{C}(I) = \{p : p \text{ is a finite partial function from } I \times \omega \text{ into } 2\}$ , ordered by  $p \leq q$  iff  $p \supseteq q$ .

*Proof.* First note that  $\langle \langle r_{i,\sigma} : i, \sigma < \kappa \rangle, \langle r'_\sigma : \sigma < \kappa \rangle \rangle$  is  $\mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$ -generic over  $V_1$ . By the c.c.c. of  $\mathbb{C}(\lambda)$  it suffices to show that for any countable set  $I \subseteq \lambda$ ,  $I \in V$ , the sequence  $\langle s_\alpha : \alpha \in I \rangle$  is  $\mathbb{C}(I)$ -generic over  $V$ . Thus it suffices to prove the following

For every  $(p, q) \in \mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$  and every open dense subset  $D \in V$

(\*) of  $\mathbb{C}(I)$ , there is  $(\bar{p}, \bar{q}) \leq (p, q)$  such that  $(\bar{p}, \bar{q}) \Vdash \langle s_\alpha : \alpha \in I \rangle$  extends some element of  $D^\top$ .

<sup>3</sup> To see this let  $\mathcal{F} = \bigcup_{n < \omega} \mathcal{F}_n$ , where  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$  and  $|\mathcal{F}_n| < \kappa_{n+1}$ , and define  $f$  so that  $\sup\{g(n) : g \in \mathcal{F}_n\} < f(n) \in \kappa_{n+1} \setminus \kappa_n$ .

<sup>4</sup> Let  $f_0$  be arbitrary. Given  $\alpha < \kappa^+$ , we can apply the above to find  $f_\alpha$  so that  $f_\alpha(n) > f_\beta(n)$ , for all large  $n$ , and all  $\beta < \alpha$ .

Let  $(p, q)$  and  $D$  be as above and for simplicity suppose that  $p = q = \emptyset$ . Let  $b \in D$ , and let  $\alpha_1, \dots, \alpha_m$  be an enumeration of the components of  $b$ , i.e., those  $\alpha$  such that  $(\alpha, n) \in \text{dom}(b)$  for some  $n$ . Also let  $\sigma_1, \dots, \sigma_m < \kappa$  be such that  $\alpha_i \in S_{\sigma_i}, i = 1, \dots, m$ . By (d) each  $I \cap S_{\sigma_i}$  is finite, thus by (c) we can find  $n^* < \omega$  such that for all  $n \geq n^*, 1 \leq i \leq m$  and  $\alpha_1^* < \alpha_2^*$  in  $I \cap S_{\sigma_i}$  we have  $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$ . Let

$$\bar{q} = \{\langle \sigma_i, n, 0 \rangle : 1 \leq i \leq m, n < n^*\}.$$

Then  $\bar{q} \in \mathbb{C}(\kappa)$  and  $(\emptyset, \bar{q}) \Vdash^{\neg} k_{\alpha_i} \geq n^{*\neg}$  for all  $1 \leq i \leq m$ . Let

$$\bar{p} = \{\langle f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0, b(\alpha_i, n) \rangle : 1 \leq i \leq m, (\alpha_i, n) \in \text{dom}(b)\}.$$

Then  $\bar{p} \in \mathbb{C}(\kappa \times \kappa)$  is well-defined and for  $(\alpha_i, n) \in \text{dom}(b), 1 \leq i \leq m$  we have

$$(\bar{p}, \bar{q}) \Vdash^{\neg} \mathfrak{L}_{\alpha_i}(n) = \mathfrak{L}_{f_{\alpha_i}(n+k_{\alpha_i}), \sigma_i}(0) = \bar{p}(f_{\alpha_i}(n+k_{\alpha_i}), \sigma_i, 0) = b(\alpha_i, n)^{\neg}$$

and hence

$$(\bar{p}, \bar{q}) \Vdash^{\neg} \langle \mathfrak{L}_{\alpha} : \alpha \in I \rangle \text{ extends } b^{\neg}.$$

(\*) follows and we are done. □

The theorem follows. □

### 3. GETTING RESULTS FROM OPTIMAL HYPOTHESES

**Theorem 3.1.** *Suppose GCH holds and  $\kappa$  is a cardinal of countable cofinality and there are  $\kappa$ -many measurable cardinals below  $\kappa$ . Then there is a cardinal preserving not adding a real extension  $V_1$  of  $V$  in which there is a splitting  $\langle S_{\sigma} : \sigma < \kappa \rangle$  of  $\kappa^+$  into sets of size  $\kappa^+$  such that for every countable set  $I \in V$  and every  $\sigma < \kappa, |I \cap S_{\sigma}| < \aleph_0$ .*

*Proof.* Let  $X$  be a set of measurable cardinals below  $\kappa$  of size  $\kappa$  which is discrete, i.e., contains none of its limit points, and for each  $\xi \in X$  fix a normal measure  $U_{\xi}$  on  $\xi$ . For each  $\xi \in X$  let  $\mathbb{P}_{\xi}$  be the Prikry forcing associated with the measure  $U_{\xi}$  and let  $\mathbb{P}_X$  be the Magidor iteration of  $\mathbb{P}_{\xi}$ 's,  $\xi \in X$  (cf. [2, 5]). Since  $X$  is discrete, each condition in  $\mathbb{P}_X$  can be seen as  $p = \langle \langle s_{\xi}, A_{\xi} \rangle : \xi \in X \rangle$  where for  $\xi \in X, \langle s_{\xi}, A_{\xi} \rangle \in \mathbb{P}_{\xi}$  and  $\text{supp}(p) = \{\xi \in X : s_{\xi} \neq \emptyset\}$  is finite. We may further suppose that for each  $\xi \in X$  the Prikry sequence for  $\xi$  is contained in  $(\text{sup}(X \cap \xi), \xi)$ . Let  $G$  be  $\mathbb{P}_X$ -generic over  $V$ . Note that  $G$  is uniquely determined by a

sequence  $(x_\xi : \xi \in X)$ , where each  $x_\xi$  is an  $\omega$ -sequence cofinal in  $\xi$ ,  $V$  and  $V[G]$  have the same cardinals, and  $GCH$  holds in  $V[G]$ .

Work in  $V[G]$ . We now force  $\langle S_\sigma : \sigma < \kappa \rangle$  as follows. The set of conditions  $\mathbb{P}$  consists of pairs  $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in V[G]$  such that:

- (1)  $\tau < \kappa^+$ ,
- (2)  $\langle s_\sigma : \sigma < \kappa \rangle$  is a splitting of  $\tau$ ,
- (3) for every countable set  $I \in V$  and every  $\sigma < \kappa$ ,  $|I \cap s_\sigma| < \aleph_0$ .

**Remark 3.2.** (a) Given a condition  $p \in \mathbb{P}$  as above,  $p$  decides an initial segment of  $S_\sigma$ , namely  $S_\sigma \cap \tau$ , to be  $s_\sigma$ . Condition (3) guarantees that each component in this initial segment has finite intersection with countable sets from the ground model.

(b) Let  $t_0 = \bigcup_{\xi \in X} x_\xi$ . By genericity arguments, it is easily seen that  $t_0$  is a subset of  $\kappa$  of size  $\kappa$  such that for all countable sets  $I \in V$ ,  $|I \cap t_0| < \aleph_0$ . For each  $i < \kappa$  set  $t_i = t_0 + i = \{\alpha + i : \alpha \in t_0\}$ . Then again by genericity arguments, for every countable set  $I \in V$ ,  $|I \cap t_i| < \aleph_0$ . Define  $s_i, i < \kappa$  by recursion as  $s_0 = t_0$  and  $s_i = t_i \setminus \bigcup_{j < i} t_j$  for  $i > 0$ . Then  $p = \langle \kappa, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in \mathbb{P}$  (since again by genericity arguments,  $\langle s_\sigma : \sigma < \kappa \rangle$  is a splitting of  $\kappa$ ), and hence  $\mathbb{P}$  is non-trivial.

We call  $\tau$  the height of  $p$  and denote it by  $ht(p)$ . For  $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle$  and  $q = \langle \nu, \langle t_\sigma : \sigma < \kappa \rangle \rangle$  in  $\mathbb{P}$  we define  $p \leq q$  iff

- (1)  $\tau \geq \nu$ ,
- (2) for every  $\sigma < \kappa$ ,  $s_\sigma \cap \nu = t_\sigma$ , i.e., each  $s_\sigma$  end extends  $t_\sigma$ .

**Lemma 3.3.** (a)  $\mathbb{P}$  satisfies the  $\kappa^{++}$ -c.c.,

(b)  $\mathbb{P}$  is  $< \kappa$ -distributive.

*Proof.* (a) is trivial, as  $|\mathbb{P}| \leq 2^\kappa = \kappa^+$ . For (b), fix  $\delta < \kappa$ ,  $\delta$  regular, and let  $p \in \mathbb{P}$  and  $\underset{\sim}{g} \in V[G]^\mathbb{P}$  be such that

$$p \Vdash \underset{\sim}{g} : \delta \rightarrow On^\gamma.$$

We find  $q \leq p$  which decides  $\mathcal{g}$ . Fix in  $V$  a splitting of  $\kappa$  into  $\delta$ -many sets of size  $\kappa$ ,  $\langle Z_i : i < \delta \rangle$ <sup>5</sup>. Let  $\theta$  be a large enough regular cardinal. Pick an increasing continuous sequence  $\langle M_i : i \leq \delta \rangle$  of elementary submodels of  $\langle H(\theta), \in \rangle$  of size  $\kappa$  such that<sup>6</sup>:

- (1)  $\langle M_i : i \leq \delta \rangle \in V[G]$ ,
- (2)  $p, \mathbb{P}, \mathcal{g}, \langle Z_i : i < \delta \rangle \in M_0$ ,
- (3) if  $i < \delta$  is a limit ordinal, then  $\langle M_j : j \leq i \rangle \in M_{i+1}$ ,
- (4)  $cf(M_\delta \cap \kappa^+) = \delta$ ,
- (5) if  $i$  is not a limit ordinal, then  $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$  for a measurable  $\xi_i$  of  $V$  in  $X$ ,
- (6)  $i < j \Rightarrow \xi_i < \xi_j$ ,
- (7)  $\langle M_i \cap V : i \leq \delta \rangle \in V$ .

For each non-limit  $i < \delta$ ,  $M_{i+1} \cap V$  is in  $V$  by clause (7), and so by clause (5),  $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$ , where  $\xi_i \in X$ , so we can pick in  $V$  a cofinal in  $M_{i+1} \cap \kappa^+$  sequence  $\langle \eta_\alpha^i : \alpha < \xi_i \rangle$ , where  $\eta_\alpha^i > M_i \cap \kappa^+$ , for all  $\alpha < \xi_i$ <sup>7</sup>.

Denote by  $\xi_i'$  the first element of the Prikry sequence of  $\xi_i$ . We define a descending sequence  $p_i = \langle \tau_i, \langle s_{i,\sigma} : \sigma < \kappa \rangle \rangle$  of conditions by induction as follows:

**i=0.** Set  $p_0 = p$ .

**i=j+1.** Assume  $p_j$  is constructed such that  $p_j \in M_j$  if  $j$  is not a limit ordinal, and  $p_j \in M_{j+1}$  if  $j$  is a limit ordinal and  $p_j$  decides  $\mathcal{g} \restriction j$ . Fix a bijection  $f_j : Z_j \rightarrow (ht(p_j), \eta_{\xi_j'}^j)$  in  $M_{j+1}$  and set<sup>8</sup>

<sup>5</sup> Note that this is possible, as  $\delta < \kappa$  are cardinals in  $V$ . The splitting can also be chosen in  $V[G]$ .

<sup>6</sup>Condition (5) can be guaranteed using the fact that the set  $K = \{\alpha < \kappa^+ : cf^V(\alpha) \in X\}$  is a stationary subset of  $\kappa^+$  in  $V[G]$  (given  $M_i$ , build a suitable continuous increasing chain  $\langle N_j : j < \kappa^+ \rangle$  consisting of models of size  $\kappa$ . Then  $\langle \sup(N_j \cap \kappa^+) : j < \kappa^+ \rangle$  forms a club of  $\kappa^+$ , and  $M_{i+1}$  can be chosen to be one of those  $N_j$  so that  $\sup(N_j \cap \kappa^+) \in K$ ). (7) can be guaranteed by the fact that  $\mathbb{P}_X$  satisfies the  $\kappa^+$ -c.c. and the models have size  $\kappa$  (use the fact that given any model  $N$  of size  $\kappa$ , there exists a model in  $V$  of the same size which contains  $N \cap V$ ).

<sup>7</sup>Note that  $\sup(M_{i+1} \cap \kappa^+) = M_{i+1} \cap \kappa^+$ . This is because if  $\xi < \kappa^+$ , and  $\xi \in M_{i+1}$ , then since  $\kappa \cup \{\kappa\} \subseteq M_{i+1}$ , and  $M_{i+1} \models |\xi| = \kappa$ , we have  $\xi \subseteq M_{i+1}$ . Also, as the sequence of  $M_i$ 's is increasing continuous,  $\sup(M_i \cap \kappa^+) = M_i \cap \kappa^+$  holds for limit ordinals  $i$ .

<sup>8</sup>It is easily seen by induction on  $j \leq i$  that  $ht(p_j) < \eta_{\xi_j'}^j$ : if  $j = 0$  or  $j$  is a successor ordinal, then  $p_j \in M_j$ , so  $ht(p_j) \in M_j \cap \kappa^+ < \eta_{\xi_j'}^j$ . If  $j$  is a limit ordinal, then  $ht(p_j) = \sup_{k < j} ht(p_k) \leq \sup_{k < j} M_k \cap \kappa^+ = M_j \cap \kappa^+ < \eta_{\xi_j'}^j$ .

$$p'_{j+1} = \langle \eta_{\xi'_j}^j, \langle s_{j,\sigma} \cup \{f_j(\sigma)\} : \sigma \in Z_j \rangle \smallfrown \langle s_{j,\sigma} : \sigma \in \kappa \setminus Z_j \rangle \rangle$$

Clearly  $p'_{j+1} \in M_{j+1}$ . Let  $p_{j+1} \in M_{j+1}$  be an extension of  $p'_{j+1}$  which decides  $g(j)$ .

**limit(i).** Let  $p_i = \langle \sup_{j < i} ht(p_j), \langle \bigcup_{j < i} s_{j,\sigma} : \sigma < \kappa \rangle \rangle$ .

Let us show that the above sequence is well-defined. Thus we need to show that for each  $i \leq \delta$ ,  $p_i \in \mathbb{P}$ . We prove this by induction on  $i$ . The successor case is trivial. Thus fix a limit ordinal  $i \leq \delta$ . If  $p_i \notin \mathbb{P}$ , we can find a countable set  $I \in V$ ,  $I \subseteq \kappa^+$ , and  $\sigma < \kappa$  such that  $I \cap s_{i,\sigma}$  is infinite. Define the sequence  $\langle \alpha(j) : j < i \rangle$  as follows:

- if  $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$ , then  $\alpha(j) \in [\sup(X \cap \xi_j), \xi_j]$  is the least such that  $\eta_{\alpha(j)}^j > \sup(I \cap (M_{j+1} \setminus M_j))$ ,
- $\alpha(j) = \sup(X \cap \xi_j)$  otherwise. Note that in this case  $\alpha(j) < \xi'_j$  (because the Prikry sequence for  $\xi$  was chosen in the interval  $(\sup(X \cap \xi), \xi)$ ).

Clearly  $\langle \alpha(j) : j < i \rangle \in V$ .

**Lemma 3.4.** *The set  $K = \{j < i : \xi'_j \leq \alpha(j)\}$  is finite.*

*Proof.* Let  $p \in \mathbb{P}_X$ ,  $p = \langle \langle s_\xi, A_\xi \rangle : \xi \in X \rangle$ . Extend  $p$  to  $q = \langle \langle t_\xi, B_\xi \rangle : \xi \in X \rangle$  by setting

- $t_\xi = s_\xi$  and  $B_\xi = A_\xi$  for  $\xi \in \text{supp}(p)$ ,
- $t_\xi = \emptyset$  and  $B_\xi = A_\xi \setminus (\alpha(j) + 1)$ , if  $\xi = \xi_j$  (some  $j < i$ ) and  $\xi \notin \text{supp}(p)$ ,
- $t_\xi = \emptyset$  and  $B_\xi = A_\xi$ , otherwise.

Then  $q \leq p$  and  $q \Vdash^\neg K \subseteq \{j < i : \xi_j \in \text{supp}(p)\}^\neg$ , so  $q \Vdash^\neg K$  is finite  $^\neg$ .  $\square$

Take  $i_0 < i$  large enough so that no point  $\geq i_0$  is in  $K$ . Then for all  $j \geq i_0$  we have  $\xi'_j > \alpha(j)$ , hence  $\eta_{\xi'_j}^j > \sup(I \cap (M_{j+1}))$  <sup>9</sup>.

**Claim 3.5.** *We have*

$$I \cap s_{i,\sigma} \subseteq I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$$

where  $i_1$  is the unique ordinal less than  $\delta$  so that  $\sigma \in Z_{i_1}$ .

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<sup>9</sup>This is trivial if  $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$ , as then  $\eta_{\xi'_j}^j > \eta_{\alpha(j)}^j > \sup(I \cap (M_{j+1} \setminus M_j)) = \sup(I \cap M_{j+1})$ . If  $I \cap (M_{j+1} \setminus M_j) = \emptyset$ , then  $\eta_{\xi'_j}^j > M_j \cap \kappa^+ = \sup(M_j \cap \kappa^+) \geq \sup(I \cap M_j)$  (as  $I \subseteq \kappa^+$ ) and  $\sup(I \cap M_{j+1}) = \sup(I \cap M_j)$  (since  $I$  has no points in  $M_{j+1} \setminus M_j$ ), and hence again  $\eta_{\xi'_j}^j > \sup(I \cap (M_{j+1}))$ .

*Proof.* Assume towards a contradiction that the inclusion fails, and let  $t \in I \cap s_{i,\sigma}$  be such that  $t \notin I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$ . As  $i$  is a limit ordinal,  $I \cap s_{i,\sigma} = I \cap \bigcup_{j < i} s_{j,\sigma}$ . Let  $j < i$  be the least such that  $t \in s_{j+1,\sigma}$ . Then as  $t \in I \cap M_{j+1}$  and  $j \geq i_0$  we have  $t < \eta_{\xi_j}^j$ , so that by our definition of  $p'_{j+1}$ ,  $t$  must be of the form  $f_j(\sigma)$ , where  $\sigma \in Z_j$ . But then  $j = i_1$  and hence  $t = f_{i_1}(\sigma)$ . This is a contradiction, and the result follows.  $\square$

Thus, as  $I \cap s_{i,\sigma}$  is infinite, we must have  $I \cap s_{i_0,\sigma}$  is also infinite, and this is in contradiction with our inductive assumption.

It then follows that  $q = p_\delta \in \mathbb{P}$  and it decides  $\underline{g}$ .  $\square$

Let  $H$  be  $\mathbb{P}$ -generic over  $V[G]$  and set  $V_1 = V[G][H]$ . It follows from Lemma 3.3 that all cardinals  $\leq \kappa$  and  $\geq \kappa^{++}$  are preserved. Also note that  $\kappa^+$  is preserved, as otherwise it would have cofinality less than  $\kappa$ , which is impossible by the  $< \kappa$ -distributivity of  $\mathbb{P}$ . Hence  $V_1$  is a cardinal preserving and not adding reals forcing extension of  $V[G]$  and hence of  $V$ . For  $\sigma < \kappa$  set  $S_\sigma = \bigcup_{\langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H} s_\sigma$ .

**Lemma 3.6.** *The sequence  $\langle S_\sigma : \sigma < \kappa \rangle$  is as required.*

*Proof.* For each  $\tau < \kappa^+$ , it is easily seen that the set of all conditions  $p$  such that  $ht(p) \geq \tau$  is dense, so  $\langle S_\sigma : \sigma < \kappa \rangle$  is a partition of  $\kappa^+$ . Now suppose that  $I \in V$  is a countable subset of  $\kappa^+$ . Find  $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H$  such that  $\tau \supseteq I$ . Then for all  $\sigma < \kappa$ ,  $S_\sigma \cap I = s_\sigma \cap I$ , and hence  $|S_\sigma \cap I| = |s_\sigma \cap I| < \aleph_0$ .  $\square$

Theorem 3.1 follows.  $\square$

**Remark 3.7.** (a) *The size of a set  $I$  in  $V$  can be changed from countable to any fixed  $\eta < \kappa$ . Given such  $\eta$ , we start with the Magidor iteration of Prikry forcings above  $\eta$ .<sup>10</sup> The rest of the conclusions are the same.*

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<sup>10</sup>The reason for starting the iteration above  $\eta$  is to add no subsets of  $\eta$ . This will guarantee that if  $t_0$  is defined as in Remark 3.2(b), then  $t_0$  has finite intersection with sets from  $V$  of size  $\eta$ . Using this fact we can show as before that there is a splitting of  $\kappa$  into  $\kappa$  sets, each of them having finite intersection with ground model sets of size  $\eta$ . This makes the second step of the above forcing construction well-behaved.

(b) *It is possible to add a one element Prikry sequence to each  $\xi \in X$ .<sup>11</sup> Then  $V_1$  will be a cofinality preserving generic extension of  $V$ .*

The next corollary follows from Theorem 3.1 and Remark 2.2.

**Corollary 3.8.** *Suppose that GCH holds in  $V$ ,  $\kappa$  is a cardinal of countable cofinality and there are  $\kappa$ -many measurable cardinals below  $\kappa$ . Then there is a cardinal preserving not adding a real extension  $V_1$  of  $V$  such that adding  $\kappa$ -many Cohen reals over  $V_1$  produces  $\kappa^+$ -many Cohen reals over  $V$ .*

**Theorem 3.9.** *Assume that there is no sharp for a strong cardinal. Suppose  $V_1 \subseteq V_2$  have the same cardinals, same reals and there is an infinite set of ordinals  $S$  in  $V_2$  which does not contain an infinite subset which is in  $V_1$ . Then either*

(1)  *$S$  is countable, and then there is a measurable cardinal  $\leq \sup(S)$  in  $\mathcal{K}$ ,*

*or*

(2)  *$S$  is uncountable, and then there is  $\delta \leq \sup(S)$  which is a limit of  $|S|$ -many or  $\delta$ -many measurable cardinals of  $\mathcal{K}$ .*

*Proof.* Given a model  $V$ , let  $\mathcal{K}(V)$  denote the core model of  $V$  below the strong cardinal. Note that  $\mathcal{K}(V_1) = \mathcal{K}(V_2)$ , since the models  $V_1$  and  $V_2$  agree about cardinals. We denote this common core model by  $\mathcal{K}$ .

Let us first assume that  $S$  is countable. Suppose otherwise, i.e., there are no measurable cardinals  $\leq \sup(S)$  in  $\mathcal{K}$ . Then by the Covering Theorem (see [6]) there is  $Y \in \mathcal{K}$ ,  $|Y| = \aleph_1$  which covers  $S$ . Fix some  $f : \aleph_1 \leftrightarrow Y$  in  $V_1$ . Consider  $Z = f^{-1} S$ . Then  $Z$  also does not contain an infinite subset which is in  $V_1$ . But  $Z$  is countable, hence there is  $\eta < \omega_1$  with  $Z \subseteq \eta$ . Let  $g : \omega \leftrightarrow \eta$  in  $V_1$ . Consider  $X = g^{-1} Z$ . Then  $X$  also does not contain an infinite subset which is in  $V_1$ . But this is impossible since  $V_1, V_2$  have the same reals (and hence  $X$  itself is in  $V_1$ ). Contradiction.

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<sup>11</sup>Conditions in the forcing are of the form  $\langle p_\xi : \xi \in X \rangle$ , where for each  $\xi \in X$ ,  $p_\xi$  is either of the form  $A_\xi$  for some  $A_\xi \in U_\xi$ , or  $\alpha_\xi$  for some  $\alpha_\xi < \xi$ . We also require that there are only finitely many  $p_\xi$ 's of the form  $\alpha_\xi$ . When extending a condition, we allow either  $A_\xi$  to become thinner, or replace it by some ordinal  $\alpha_\xi \in A_\xi$ .



Let us deal now with the uncountable case. Suppose otherwise, i.e., there is no  $\delta \leq \sup(S)$  which is a limit of  $|S|$ -many or  $\delta$ -many measurable cardinals of  $\mathcal{K}$ . Pick a counterexample  $S$  with  $\sup(S)$  as small as possible. Denote  $\sup(S)$  by  $\delta$ . By minimality,  $\delta$  is a cardinal. Also, the measurable cardinals of  $\mathcal{K}$  are unbounded in  $\delta$ . For otherwise, let  $\xi$  be their supremum. Pick  $S' \subseteq S$  of size  $\xi$ . By the Covering Theorem,  $S'$  can be covered by a set in  $\mathcal{K}$  of size  $\xi < \delta$ , and then we get a contradiction to the minimality of  $\delta$ , as witnessed by  $\xi$  and  $S'$  <sup>12</sup>.

Clearly,  $\delta$  must be a singular cardinal and by the above,  $\delta$  is a limit of measurable cardinals in  $\mathcal{K}$ . Fix a cofinal sequence  $\langle \delta_i : i < \text{cf}(\delta) \rangle$ . Denote by  $\eta$  the cardinality of the set  $\{\alpha < \delta : \alpha \text{ is a measurable cardinal in } \mathcal{K}\}$ . By the assumption,  $|S| > \eta \geq \text{cf}(\delta)$ . But then there is  $i^* < \text{cf}(\delta)$  such that  $S \cap \delta_{i^*}$  has size  $> \eta$ . This is impossible by the minimality of  $\delta$ . Contradiction.  $\square$

The conclusions of the theorem are optimal. A Prikry sequence witnesses this in the countable case and the Magidor iteration of Prikry forcing witnesses this in the uncountable case.

**Theorem 3.10.** *Suppose that  $V_1 \supseteq V$  are such that:*

- (a)  $V_1$  and  $V$  have the same cardinals and reals,
- (b)  $\kappa < \lambda$  are infinite cardinals of  $V_1$ ,
- (c) there is no splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\lambda$  in  $V_1$  as in Theorem 2.1(d).

*Then adding  $\kappa$ -many Cohen reals over  $V_1$  cannot produce  $\lambda$ -many Cohen reals over  $V$ .*

*Proof.* Suppose not. Let  $\langle r_\alpha : \alpha < \lambda \rangle$  be a sequence of  $\lambda$ -many Cohen reals over  $V$  added after forcing with  $\mathbb{C}(\kappa)$  over  $V_1$ . Let  $G$  be  $\mathbb{C}(\kappa)$ -generic over  $V_1$ . For each  $p \in \mathbb{C}(\kappa)$  set

$$C_p = \{\alpha < \lambda : p \text{ decides } \mathcal{I}_\alpha(0)\}.$$

Then by genericity  $\lambda = \bigcup_{p \in G} C_p$ . Fix an enumeration  $\langle p_\xi : \xi < \kappa \rangle$  of  $G$ , and define a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\lambda$  in  $V_1[G]$  by setting  $S_\sigma = C_{p_\sigma} \setminus \bigcup_{\xi < \sigma} C_{p_\xi}$ . By (a) and (c) we

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<sup>12</sup>We then have  $\sup(S') = \xi < \delta$  and  $S'$  is a counterexample to our assumption of smaller supremum.

can find a countable  $I \in V$  and  $\sigma < \kappa$  such that  $I \subseteq S_\sigma$ .<sup>13</sup> Suppose for simplicity that  $\forall \alpha \in S_\sigma, p_\sigma \Vdash \check{r}_\alpha(0) = 0^\top$ . Let  $q \in \mathbb{C}(\kappa)$  be such that

$$q \Vdash^V I \in V \text{ is countable and } \forall \alpha \in I, \check{r}_\alpha(0) = 0^\top.$$

Pick  $\langle 0, \alpha \rangle \in \omega \times I$  such that  $\langle 0, \alpha \rangle \notin \text{supp}(q)$ . Let  $\bar{q} = q \cup \{\langle \langle 0, \alpha \rangle, 1 \rangle\}$ . Then  $\bar{q} \in \mathbb{C}(\kappa)$ ,  $\bar{q} \leq q$  and  $\bar{q} \Vdash \check{r}_\alpha(0) = 1^\top$ , which is a contradiction.  $\square$

The following corollary answers a question from [1].

**Corollary 3.11.** *The following are equiconsistent:*

- (a) *There exists a pair  $(V_1, V_2)$ ,  $V_1 \subseteq V_2$  of models of set theory with the same cardinals and reals and a cardinal  $\kappa$  of cofinality  $\omega$  (in  $V_2$ ) such that adding  $\kappa$ -many Cohen reals over  $V_2$  adds more than  $\kappa$ -many Cohen reals over  $V_1$ .*
- (b) *There exists a cardinal  $\delta$  which is a limit of  $\delta$ -many measurable cardinals.*

*Proof.* Assume (a) holds for some pair  $(V_1, V_2)$  of models of set theory,  $V_1 \subseteq V_2$  which have the same cardinals and reals. If there is a sharp for a strong cardinal, then clearly in  $\mathcal{K}$ , the core model for a strong cardinal, there is a cardinal  $\delta$  which is a limit of  $\delta$ -many measurable cardinals<sup>14</sup>. So assume there is no sharp for a strong cardinal. Then by Theorem 3.10 there exists a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\kappa^+$  in  $V_2$  such that for every countable set  $I \in V_1$  and  $\sigma < \kappa$ ,  $I \cap S_\sigma$  is finite. Take  $S$  to be one of the sets  $S_\sigma$  which has size  $\kappa^+$ . So by Theorem 3.9, we get the consistency of (b)<sup>15</sup>.

Conversely if (b) is consistent, then by Corollary 3.8 the consistency of (a) follows<sup>16</sup>.  $\square$

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<sup>13</sup>In fact, by (c) there exist a countable  $I \in V$  and some  $\sigma < \kappa$  such that  $I \cap S_\sigma$  is infinite. By (a),  $V$  and  $V_1$  have the same reals, and hence  $I \cap S_\sigma \in V$ . So by replacing  $I$  with  $I \cap S_\sigma$ , if necessary, we can assume that  $I \subseteq S_\sigma$ .

<sup>14</sup>In fact there are many such cardinals  $\delta$ .

<sup>15</sup>Note that necessarily case (b) of Theorem 3.9 happens.

<sup>16</sup>If  $\text{cf}(\delta) > \omega$ , then we can find  $\delta^* < \delta$  of cofinality  $\omega$  which is a limit of  $\delta^*$ -many measurable cardinals, so that Corollary 3.8 can be applied. To see such a  $\delta^*$  exists, define an increasing sequence  $\delta_n, n < \omega$ , of cardinals below  $\delta$ , so that for any  $n$ , there are at least  $\delta_n$ -many measurable cardinals below  $\delta_{n+1}$ , and let  $\delta^* = \sup_n \delta_n$ .

4. BELOW THE FIRST FIXED POINT OF THE  $\aleph$ -FUNCTION

**Theorem 4.1.** *Suppose that  $V_1 \supseteq V$  are such that  $V_1$  and  $V$  have the same cardinals and reals. Suppose  $\aleph_\delta <$  the first fixed point of the  $\aleph$ -function,  $X \subseteq \aleph_\delta, X \in V_1$  and  $|X| \geq \delta^+$  (in  $V_1$ ). Then  $X$  has a countable subset which is in  $V$ .*

*Proof.* By induction on  $\delta <$  the first fixed point of the  $\aleph$ -function.

**Case 1.**  $\delta = 0$ . Then  $X \in V$  by the fact that  $V_1$  and  $V$  have the same reals.

**Case 2.**  $\delta = \delta' + 1$ . We have  $\delta' < \aleph_{\delta'}$ , hence  $\delta^+ < \aleph_\delta$ , thus we may suppose that  $|X| \leq \aleph_{\delta'}$ . Let  $\eta = \sup(X) < \aleph_\delta$ . Pick  $f_\eta : \aleph_{\delta'} \leftrightarrow \eta, f_\eta \in V$ . Set  $Y = f_\eta^{-1} X$ . Then  $Y \subseteq \aleph_{\delta'}, \delta' < \aleph_{\delta'}$  and  $|Y| \geq \delta^+ = \delta'^+$ . Hence by induction there is a countable set  $B \in V$  such that  $B \subseteq Y$ . Let  $A = f_\eta'' B$ . Then  $A \in V$  is a countable subset of  $X$ .

**Case 3.**  $\text{limit}(\delta)$ . Let  $\langle \delta_\xi : \xi < cf\delta \rangle$  be increasing and cofinal in  $\delta$ . Pick  $\xi < cf\delta$  such that  $|X \cap \aleph_{\delta_\xi}| \geq \delta^+$ . By induction there is a countable set  $A \in V$  such that  $A \subseteq X \cap \aleph_{\delta_\xi} \subseteq X$ .  $\square$

The following corollary gives a negative answer to another question from [1].

**Corollary 4.2.** *Suppose  $V_1, V$  and  $\delta$  are as in Theorem 4.1. Then adding  $\aleph_\delta$ -many Cohen reals over  $V_1$  cannot produce  $\aleph_{\delta+1}$ -many Cohen reals over  $V$ .*

*Proof.* Towards a contradiction suppose that adding  $\aleph_\delta$ -many Cohen reals over  $V_1$  produces  $\aleph_{\delta+1}$ -many Cohen reals over  $V$ . Then by Theorem 3.10, there exists  $X \subseteq \aleph_{\delta+1}, X \in V_1$  such that  $|X| = \aleph_{\delta+1} (\geq \delta^+)$  and  $X$  does not contain any countable subset from  $V$ <sup>17</sup>, which is in contradiction with Theorem 4.1.  $\square$

 5. AT THE FIRST FIXED POINT OF THE  $\aleph$ -FUNCTION

The next theorem shows that Theorem 4.1 does not extend to the first fixed point of the  $\aleph$ -function.

**Theorem 5.1.** *Suppose GCH holds and  $\kappa$  is the least singular cardinal of cofinality  $\omega$  which is a limit of  $\kappa$ -many measurable cardinals. Then there is a pair  $(V[G], V[H])$  of generic extensions of  $V$  with  $V[G] \subseteq V[H]$  such that:*

<sup>17</sup>In fact, there exists a splitting  $\langle S_\sigma : \sigma < \aleph_\delta \rangle$  of  $\aleph_{\delta+1}$  in  $V_1$ , consisting of sets of size  $\aleph_{\delta+1}$  such that each  $S_\sigma$  has finite intersection with any countable set from  $V$ . The set  $X$  can be chosen to be any of  $S_\sigma$ 's.

- (a)  $V[G]$  and  $V[H]$  have the same cardinals and reals,
- (b)  $\kappa$  is the first fixed point of the  $\aleph$ -function in  $V[G]$  ( and hence in  $V[H]$ ),
- (c) in  $V[H]$  there exists a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\kappa$  into sets of size  $\kappa$  such that for every countable  $I \in V[G]$  and  $\sigma < \kappa$ ,  $|I \cap S_\sigma| < \aleph_0$ .

*Proof.* We first give a simple observation.

**Claim 5.2.** *Suppose there is  $S \subseteq \kappa$  of size  $\kappa$  in  $V[H] \supseteq V[G]$  such that for every countable  $A \in V[G]$ ,  $|A \cap S| < \aleph_0$ . Then there is a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\kappa$  as in (c).*

*Proof.* Let  $\langle \alpha_i : i < \kappa \rangle$  be an increasing enumeration of  $S$ . We may further suppose that  $\alpha_0 = 0$ , each  $\alpha_i, i > 0$  is measurable<sup>18</sup> in  $V$  and is not a limit point of  $S$ .<sup>19</sup> Note that for all  $i < \kappa$ ,  $\sup_{j < i} \alpha_j < \alpha_i \setminus \sup_{j < i} \alpha_j$ . Now set:

$$S_0 = S,$$

$$S_\sigma = \{\alpha_l + \sigma : i \leq l < \kappa\}, \text{ for } 0 < \sigma \in [\sup_{j < i} \alpha_j, \alpha_i).$$

Then  $\langle S_\sigma : \sigma < \kappa \rangle$  is as required (note that for  $\sigma > 0$ ,  $S_\sigma \subseteq S + \sigma = \{\alpha + \sigma : \alpha \in S\}$ , and clearly  $S + \sigma$ , and hence  $S_\sigma$ , has finite intersection with countable sets from  $V[G]$ ).  $\square$

Thus it is enough to find a pair  $(V[G], V[H])$  of generic extensions of  $V$  satisfying (a) and (b) with  $V[G] \subseteq V[H]$  such that in  $V[H]$  there is  $S \subseteq \kappa$  of size  $\kappa$  composed of inaccessibles, such that for every countable  $A \in V[G]$ ,  $|A \cap S| < \aleph_0$ .

Let  $X$  be a discrete set of measurable cardinals below  $\kappa$  of size  $\kappa$ , and for each  $\xi \in X$  fix a normal measure  $U_\xi$  on  $\xi$ . For each  $\xi \in X$  we define two forcing notions  $\mathbb{P}_\xi$  and  $\mathbb{Q}_\xi$  as follows.

**Remark 5.3.** *In the following definitions we let  $\sup(X \cap \xi) = \omega$  for  $\xi = \min X$ .*

A condition in  $\mathbb{P}_\xi$  is of the form  $p = \langle s_\xi, A_\xi, f_\xi \rangle$  where

- (1)  $s_\xi \in [\xi \setminus \sup(X \cap \xi)]^{<2}$ ,
- (2) if  $s_\xi \neq \emptyset$  then  $s_\xi(0)$  is an inaccessible cardinal,

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<sup>18</sup>In fact it suffices for each  $\alpha_i$  to be inaccessible in  $V$ .

<sup>19</sup>Let  $f \in V$  be such that  $f : \kappa \rightarrow X$  is a bijection, where  $X$  is a discrete set of measurable cardinals of  $V$  below  $\kappa$  of size  $\kappa$ . Then if  $S \subseteq \kappa$  satisfies the claim, so does  $f[S]$ , hence we can suppose all non-zero elements of  $S$  are measurable in  $V$ , and are not a limit point of  $S$ .

- (3)  $A_\xi \in U_\xi$ ,
- (4)  $\max s_\xi < \min A_\xi$ ,
- (5)  $s_\xi = \emptyset \Rightarrow f_\xi \in \text{Col}(\sup(X \cap \xi)^+, < \xi)$ , where  $\text{Col}(\sup(X \cap \xi)^+, < \xi)$  is the Lévy collapse for collapsing all cardinals less than  $\xi$  to  $\sup(X \cap \xi)^+$ , and making  $\xi$  become the successor of  $\sup(X \cap \xi)^+$ ,
- (6)  $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$  where  $f_\xi^1 \in \text{Col}(\sup(X \cap \xi)^+, < s_\xi(0))$  and  $f_\xi^2 \in \text{Col}((s_\xi(0))^+, < \xi)$ .

For  $p, q \in \mathbb{P}_\xi$ ,  $p = \langle s_\xi, A_\xi, f_\xi \rangle$  and  $q = \langle t_\xi, B_\xi, g_\xi \rangle$  we define  $p \leq q$  iff

- (1)  $s_\xi$  end extends  $t_\xi$ ,
- (2)  $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$ ,
- (3)  $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$ ,
- (4)  $t_\xi = \emptyset$  and  $s_\xi \neq \emptyset \Rightarrow \sup(\text{ran}(g_\xi)) < s_\xi(0)$  and  $f_\xi^1 \leq g_\xi$ ,
- (5)  $t_\xi \neq \emptyset \Rightarrow f_\xi^1 \leq g_\xi^1$  and  $f_\xi^2 \leq g_\xi^2$  (note that in this case we have  $s_\xi = t_\xi$ ).

We also define  $p \leq^* q$  ( $p$  is a Prikry or a direct extension of  $q$ ) iff

- (1)  $p \leq q$ ,
- (2)  $s_\xi = t_\xi$ .

The proof of the following lemma is essentially the same as in the proofs in [2, 5].

**Lemma 5.4.** *(GCH) (a)  $\mathbb{P}_\xi$  satisfies the  $\xi^+ - \text{c.c.}$*

*(b) Suppose  $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$  and  $l(s_\xi) = 1$  (where  $l(s_\xi)$  is the length of  $s_\xi$ ). Then  $\mathbb{P}_\xi/p = \{q \in \mathbb{P}_\xi : q \leq p\}$  satisfies the  $\xi - \text{c.c.}$*

*(c)  $(\mathbb{P}_\xi, \leq, \leq^*)$  satisfies the Prikry property, i.e., given  $p \in \mathbb{P}$  and a sentence  $\sigma$  of the forcing language for  $(\mathbb{P}, \leq)$ , there exists  $q \leq^* p$  which decides  $\sigma$ .*

*(d) Let  $G_\xi$  be  $\mathbb{P}_\xi$ -generic over  $V$  and let  $\langle s_\xi(0) \rangle$  be the one element sequence added by  $G_\xi$ . Then in  $V[G_\xi]$ , GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals  $(\sup(X \cap \xi)^{++}, s_\xi(0))$  and  $(s_\xi(0)^{++}, \xi)$ , which are collapsed to  $\sup(X \cap \xi)^+$  and  $s_\xi(0)^+$  respectively.*

We now define the forcing notion  $\mathbb{Q}_\xi$ . A condition in  $\mathbb{Q}_\xi$  is of the form  $p = \langle s_\xi, A_\xi, f_\xi \rangle$  where

- (1)  $s_\xi \in [\xi \setminus \sup(X \cap \xi)^+]^{<3}$ ,

- (2) if  $s_\xi \neq \emptyset$  then for all  $i < l(s_\xi)$ ,  $s_\xi(i)$  is an inaccessible cardinal,
- (3)  $A_\xi \in U_\xi$ ,
- (4)  $\max s_\xi < \min A_\xi$ ,
- (5)  $s_\xi = \emptyset \Rightarrow f_\xi \in \text{Col}(\sup(X \cap \xi)^+, < \xi)$ ,
- (6)  $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$  where,  $f_\xi^1 \in \text{Col}(\sup(X \cap \xi)^+, < s_\xi(0))$  and  $f_\xi^2 \in \text{Col}((s_\xi(0))^+, < \xi)$ .

For  $p, q \in \mathbb{Q}_\xi$ ,  $p = \langle s_\xi, A_\xi, f_\xi \rangle$  and  $q = \langle t_\xi, B_\xi, g_\xi \rangle$  we define  $p \leq q$  iff

- (1)  $s_\xi$  end extends  $t_\xi$ ,
- (2)  $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$ ,
- (3)  $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$ ,
- (4)  $t_\xi = \emptyset$  and  $s_\xi \neq \emptyset \Rightarrow \sup(\text{ran}(g_\xi)) < s_\xi(0)$  and  $f_\xi^1 \leq g_\xi$ ,
- (5)  $t_\xi \neq \emptyset$  and  $s_\xi = t_\xi \Rightarrow f_\xi^1 \leq g_\xi^1$  and  $f_\xi^2 \leq g_\xi^2$ ,
- (6)  $t_\xi \neq \emptyset$  and  $s_\xi \neq t_\xi \Rightarrow \sup(\text{ran}(g_\xi^2)) < s_\xi(1)$ ,  $f_\xi^1 \leq g_\xi^1$  and  $f_\xi^2 \leq g_\xi^2$ .

We also define  $p \leq^* q$  iff

- (1)  $p \leq q$ ,
- (2)  $s_\xi = t_\xi$ .

As above we have the following.

**Lemma 5.5.** *(GCH) (a)  $\mathbb{Q}_\xi$  satisfies the  $\xi^+ - \text{c.c.}$*

*(b) Suppose  $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi$ ,  $l(s_\xi) = 2$ . Then  $\mathbb{Q}_\xi/p = \{q \in \mathbb{Q}_\xi : q \leq p\}$  satisfies the  $\xi - \text{c.c.}$ .*

*(c)  $(\mathbb{Q}_\xi, \leq, \leq^*)$  satisfies the Prikry property.*

*(d) Let  $H_\xi$  be  $\mathbb{Q}_\xi$ -generic over  $V$  and let  $\langle s_\xi(0), s_\xi(1) \rangle$  be the two element sequence added by  $H_\xi$ . Then in  $V[H_\xi]$ , GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals  $(\sup(X \cap \xi)^{++}, s_\xi(0))$  and  $(s_\xi(0)^{++}, \xi)$ , which are collapsed to  $\sup(X \cap \xi)^+$  and  $s_\xi(0)^+$  respectively.*

Now let  $\mathbb{P}$  be the Magidor iteration of the forcings  $\mathbb{P}_\xi, \xi \in X$ , and  $\mathbb{Q}$  be the Magidor iteration of the forcings  $\mathbb{Q}_\xi, \xi \in X$ . Since the set  $X$  is discrete we can view each condition in  $\mathbb{P}$  as a sequence  $p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$  where for each  $\xi \in X$ ,  $\langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$  and  $\text{supp}(p) = \{\xi : s_\xi \neq \emptyset\}$  is finite. Similarly each condition in  $\mathbb{Q}$  can be viewed as a sequence

$p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$  where for each  $\xi \in X$ ,  $\langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi$  and  $\text{supp}(p) = \{\xi : s_\xi \neq \emptyset\}$  is finite (for more information see [2, 4, 5]).

**Notation 5.6.** *If  $p$  is as above, then we write  $p(\xi)$  for  $\langle s_\xi, A_\xi, f_\xi \rangle$ .*

We also define

$$\pi : \mathbb{Q} \rightarrow \mathbb{P}$$

by

$$\pi(\langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle) = \langle \langle s_\xi \upharpoonright 1, A_\xi, f_\xi \rangle : \xi \in X \rangle.$$

It is clear that  $\pi$  is well-defined.

**Lemma 5.7.**  *$\pi$  is a projection, i.e.,*

- (a)  $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$ ,
- (b)  $\pi$  is order preserving,
- (c) if  $p \in \mathbb{Q}$ ,  $q \in \mathbb{P}$  and  $q \leq \pi(p)$  then there is  $r \leq p$  in  $\mathbb{Q}$  such that  $\pi(r) \leq q$ .

Now let  $H$  be  $\mathbb{Q}$ -generic over  $V$  and let  $G = \pi'' H$  be the filter generated by  $\pi'' H$ . Then  $G$  is  $\mathbb{P}$ -generic over  $V$ .

**Lemma 5.8.** (a) *if  $\langle \tau_\xi : \xi \in X \rangle$  and  $\langle \langle \eta_\xi^0, \eta_\xi^1 \rangle : \xi \in X \rangle$  are the Prikry sequences added by  $G$  and  $H$  respectively, then  $\tau_\xi = \eta_\xi^0$  for all  $\xi \in X$ .*

(b) *The models  $V[G]$  and  $V[H]$  satisfy the GCH, have the same cardinals and reals, and furthermore the only cardinals of  $V$  below  $\kappa$  which are preserved are  $\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}$ .*

(c)  *$\kappa$  is the first fixed point of the  $\aleph$ -function in  $V[G]$  (and hence in  $V[H]$ ).*

*Proof.* (a) and (b) follow easily from Lemmas 5.4 and 5.5 and the definition of the projection  $\pi$ . Let's prove (c). It is clear that  $\kappa$  is a fixed point of the  $\aleph$ -function in  $V[G]$ . On the other hand, by (b), the only cardinals of  $V$  below  $\kappa$  which are preserved in  $V[G]$  are  $\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}$ , and so if  $\lambda < \kappa$  is a limit cardinal in  $V[G]$ , then  $\lambda \in \text{lim}(X)$ . But by our assumption on  $\kappa$ , if  $\lambda \in \text{lim}(X)$ , then  $X \cap \lambda$  has order type less than  $\lambda$ , and hence  $(\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}) \cap \lambda$  has order type less than  $\aleph_\lambda$ . Thus  $\lambda < \aleph_\lambda$ .  $\square$

Let  $\mathbb{Q}/G = \{p \in \mathbb{Q} : \pi(p) \in G\}$ . Then  $V[H]$  can be viewed as a generic extension of  $V[G]$  by  $\mathbb{Q}/G$ .

**Lemma 5.9.**  *$\mathbb{Q}/G$  is cone homogenous: given  $p$  and  $q$  in  $\mathbb{Q}/G$  there exist  $p^* \leq p, q^* \leq q$  and an isomorphism  $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$ .*

*Proof.* Suppose  $p, q \in \mathbb{Q}/G$ . Extend  $p$  and  $q$  to  $p^* = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$  and  $q^* = \langle \langle t_\xi, B_\xi, g_\xi \rangle : \xi \in X \rangle$  respectively so that the following conditions are satisfied:

- (1)  $\text{supp}(p^*) = \text{supp}(q^*)$ . Call this common support  $K$ .
- (2) For every  $\xi \in K, l(s_\xi) = l(t_\xi) = 2$ . Note that then for every  $\xi \in K, s_\xi(0) = t_\xi(0) = \tau_\xi$ ,  $f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$  and  $g_\xi = \langle g_\xi^1, g_\xi^2 \rangle$  where  $f_\xi^1, g_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$  and  $f_\xi^2, g_\xi^2 \in \text{Col}(\tau_\xi^+, < \xi)$ .
- (3) For every  $\xi \in K, A_\xi = B_\xi$ .
- (4) For every  $\xi \in K, \text{dom}(f_\xi^1) = \text{dom}(g_\xi^1)$  and  $\text{dom}(f_\xi^2) = \text{dom}(g_\xi^2)$ .
- (5) For every  $\xi \in K$ , there exists an automorphism  $\rho_\xi^1$  of  $\text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$  such that  $\rho_\xi^1(f_\xi^1) = g_\xi^1$ .
- (6) For every  $\xi \in K$ , there exists an automorphism  $\rho_\xi^2$  of  $\text{Col}(\tau_\xi^+, < \xi)$  such that  $\rho_\xi^2(f_\xi^2) = g_\xi^2$ .

Note that clauses (5) and (6) are possible, as the corresponding forcing notions are homogeneous.

We now define  $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$  as follows. Suppose  $r \in \mathbb{Q}/G, r \leq p^*$ . Let  $r = \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \rangle$ . Then for every  $\xi \in K, r_\xi = s_\xi$ , and  $h_\xi = \langle h_\xi^1, h_\xi^2 \rangle$  where  $h_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$  and  $h_\xi^2 \in \text{Col}(\tau_\xi^+, < \xi)$ . Let

$$\rho(r) = \langle \langle t_\xi, C_\xi, \langle \rho_\xi^1(h_\xi^1), \rho_\xi^2(h_\xi^2) \rangle \rangle : \xi \in K \rangle \frown \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \setminus K \rangle.$$

It is easily seen that  $\rho$  is an isomorphism from  $(\mathbb{Q}/G)/p^*$  to  $(\mathbb{Q}/G)/q^*$ . □

The following lemma completes the proof.

**Lemma 5.10.** *Let  $S = \{\eta_\xi^1 : \xi \in X\}$ . Then  $S$  is a subset of  $\kappa$  of size  $\kappa$  and  $|A \cap S| < \aleph_0$  for every countable set  $A \in V[G]$ .*



**Remark 5.11.** (a) Since  $V[G]$  and  $V[H]$  have the same reals, it suffices to prove the lemma for  $A \subseteq S, A \in V[G]$ . In fact suppose that the lemma is true for all countable  $A \subseteq S, A \in V[G]$ . If the lemma fails, then for some countable set  $B \in V[G], |B \cap S| = \aleph_0$ . Let  $g : \omega \rightarrow B$  be a bijection in  $V[G]$ . Then  $g^{-1}[B \cap S]$  is a subset of  $\omega$  which is in  $V[H]$ , and hence in  $V[G]$ . Thus  $B \cap S \in V[G]$ . Hence we find a countable subset  $A \subseteq S$  in  $V[G]$ , namely  $B \cap S$ , for which the lemma fails, which is in contradiction with our initial assumption.

(b) In what follows we say  $A$  codes  $\xi$  (for  $\xi \in X$ ), if  $\eta_\xi^1 \in A$ .

*Proof.* Let  $\check{S}$  be a  $\mathbb{Q}/G$ -name for  $S$ . Also let  $p_0 \in H \cap \mathbb{Q}/G$  be such that  $p_0 \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A} \subseteq \check{S}$  is countable $^\top$ .

**Claim 5.12.** For every  $p \in \mathbb{Q}/G$  and every  $\xi \in X \setminus \text{supp}(p)$  there is  $q \leq p$  in  $\mathbb{Q}/G$  such that  $\xi \in \text{supp}(q)$  and if  $q(\xi) = \langle s_\xi, A_\xi, f_\xi \rangle$ , then  $l(s_\xi) = 2$  and  $q \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\top$ .

*Proof.* Let  $p$  and  $\xi$  be as in the claim. First pick  $\langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle \in G$ , and then let  $q = p \frown \langle \langle s_\xi, A_\xi, f_\xi \rangle \rangle$ , where  $s_\xi(0) = t_\xi(0) = \tau_\xi$ ,  $s_\xi(1) < \xi$  is large enough so that  $s_\xi(1) \notin A$ ,  $\text{sup}(ran(f_\xi^2)) < s_\xi(1)$  and  $s_\xi(1)$  is inaccessible. Then  $\pi(\langle \langle s_\xi, A_\xi, f_\xi \rangle \rangle) = \langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle \in G$ . On the other hand  $\pi(p) \in G$ . Let  $r \in G, r \leq \pi(p), \langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle$ . Then  $r \leq \pi(q)$ , hence  $\pi(q) \in G$ . This implies that  $q \in \mathbb{Q}/G$ . Clearly  $q$  satisfies the requirements of the Claim.  $\square$

It follows that the set

$$D = \{p \in \mathbb{Q}/G : \forall \xi \in X \setminus \text{supp}(p) \text{ there exists } q \leq p \text{ as in the above Claim}\}$$

is dense open in  $\mathbb{Q}/G$ . Let  $p \in H \cap D$ . We can assume that  $p \leq p_0$ . We show that  $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A}$  codes  $\xi$  then  $\xi \in \text{supp}(p)^\top$ . To see this suppose that  $\xi \in X \setminus \text{supp}(p)$ . Thus by Claim 5.12 we can find  $q \leq p$  in  $\mathbb{Q}/G$  such that  $\xi \in \text{supp}(q)$  and if  $q(\xi) = \langle s_\xi, A_\xi, f_\xi \rangle$ , then  $l(s_\xi) = 2$  and  $q \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\top$ . It then follows that  $\sim p \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\top$ . But then by the cone homogeneity of  $\mathbb{Q}/G$  we have  $p \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\top$ <sup>20</sup>. Hence  $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A}$  does not code  $\xi^\top$ . This means that  $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A} \subseteq \{s_\xi(1) : \xi \in \text{supp}(p)\} = \{\eta_\xi^1 : \xi \in \text{supp}(p)\}^\top$ . Lemma 5.10

<sup>20</sup>If not, then for some  $p' \leq p, p' \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\top$ . By cone homogeneity of  $\mathbb{Q}/G$  we can find  $q^* \leq q, p^* \leq p'$  and an isomorphism  $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$ . But then by standard forcing arguments and the fact that  $q^* \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\top$ , we can conclude that  $p^* \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\top$ , which is impossible, as  $p^* \leq p'$  and  $p' \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\top$ .

follows by noting that  $p \in H$  and since the Magidor iteration is used, the support of any condition is finite.  $\square$

Theorem 5.1 follows.  $\square$

The following theorem can be proved by combining the methods of the proofs of Theorems 3.1 and 5.1.

**Theorem 5.13.** *Suppose GCH holds and  $\kappa$  is the least singular cardinal of cofinality  $\omega$  which is a limit of  $\kappa$ -many measurable cardinals. Also let  $V[G]$  and  $V[H]$  be the models constructed in the proof of Theorem 5.1. Then there is a cardinal preserving, not adding a real generic extension  $V[H][K]$  of  $V[H]$  such that in  $V[H][K]$  there exists a splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  of  $\kappa^+$  into sets of size  $\kappa^+$  such that for every countable set  $I \in V[G]$  and  $\sigma < \kappa$ ,  $|I \cap S_\sigma| < \aleph_0$ .*

*Proof.* Work over  $V[H]$  and force the splitting  $\langle S_\sigma : \sigma < \kappa \rangle$  as in the proof of Theorem 3.1, with  $V, V[G]$  used there replaced by  $V[G], V[H]$  here respectively. The role of the sequence  $\bigcup_{\xi \in X} x_\xi$  in the proof of Theorem 3.1 is now played by the sequence  $S = \{\eta_\xi^1 : \xi \in X\}$ .  $\square$

**Corollary 5.14.** *Suppose GCH holds and there exists a cardinal  $\kappa$  which is of cofinality  $\omega$  and is a limit of  $\kappa$ -many measurable cardinals. Then there is pair  $(V_1, V_2)$  of models of ZFC,  $V_1 \subseteq V_2$  such that:*

- (a)  $V_1$  and  $V_2$  have the same cardinals and reals.
- (b)  $\kappa$  is the first fixed point of the  $\aleph$ -function in  $V_1$  (and hence in  $V_2$ ).
- (c) Adding  $\kappa$ -many Cohen reals over  $V_2$  adds  $\kappa^+$ -many Cohen reals over  $V_1$ .

*Proof.* Let  $V_1 = V[G]$  and  $V_2 = V[H][K]$ , where  $V[G], V[H][K]$  are as in Theorem 5.13. The result follows using Remark 2.2 and Theorem 5.13.  $\square$

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